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RANDOM BEAM PROPAGATION IN ACCELERATORS AND STORAGE RINGS

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Abstract

A kinetic equation for the joint probability distribution for fixed values of the classical action, momentum and density has been derived. Further the hydrodynamic equations of continuity and balance of momentum density have been transformed into a Schroedinger-like equation, describing particle motion in an effective electro-magnetic field and an expression for the gauge potentials has been obtained.

1 Introduction

The dynamics of particles in accelerators and storage rings is usually studied on the basis of deterministic causal tools, provided by Hamilton's equations of motion. There are, however, a number of instances where such a description fails or may not be adequate.

The beam circulating in an accelerator may be generally considered as an ensemble of nonlinear oscillators. Even in the case when the beam is not dominated by space charge these oscillators are weakly coupled at least linearly. This coupling is due to direct interaction between particles, thus revealing the discrete nature of their correlations and/or an interaction between them via the surroundings. In an experiment involving macroscopic measuring devices the observed quantities are a restricted number of variables characterizing the macroscopic state of the beam. Since the beam consists of a large number of randomly moving particles the macroscopic quantities are subject to deviations around certain mean values. These deviations appearing to an observer as random events are called fluctuations.

Therefore the particle beam propagates in response to fluctuations, automatically implied by the existence of many degrees of freedom. Fluctuating contributions remain small in comparison with the macroscopic quantities for systems in thermodynamic equilibrium [1], except at critical points. When a certain dynamic (or plasma) instability is encountered, fluctuations are expected to grow considerably. As a result the corresponding macroscopic evolution exhibits abrupt changes of various thermodynamic parameters.

In the light of the above considerations we analyze in the present paper the motion of a test particle in the bath provided by the rest of the beam. Microscopically, each particle feels the fluctuating field (due to electro-magnetic and other possible interactions) produced by all the other particles in the beam, and therefore it constantly undergoes Brownian motion in phase space. The coefficients in the resulting Fokker-Planck equation contain the fluctuation spectrum of the interparticle interactions [2, 3]. We are not going to calculate these coefficients explicitly here (we hope to do so in a forthcoming paper), but rather we study the motion of a test particle suspended in a random inhomogeneous medium under the action of external forces. The statistical properties of the medium comprising the rest of the beam are characterized by a random velocity field (which may be regarded also as a fluctuating vector electro- magnetic potential) and a random potential field. Moreover, we consider the beam fluid inviscid. The latter restriction is not essential for we presume friction and other sources of dissipation (such as synchrotron radiation) could be without effort implemented in the development presented here.

Recently a thermal wave model for relativistic charged particle beam propagation, building on remarkable analogies between particle beam optics and non relativistic quantum mechanics has been proposed [4]. The conjectured in Reference 4 Schroedinger-like equation for the transverse motion of azimuthally symmetric beams has been derived [5] in the framework of Nelson's stochastic mechanics. Further development of the model suitable to cover the problem of asymmetric beam propagation in accelerators can be found in [6]. In the present paper we recover the Nelson's scheme of stochastic mechanics for particle beams from a different point of view.

2 The Model of Random Beam Propagation

The classical motion of charged particles in an accelerator is described usually with respect to a comoving with the beam reference frame. Not taking into account chromatic effects the dynamics in the longitudinal direction can be decoupled from the dynamics in a plane transversal to the orbit. Then the evolution of the beam in 6D phase space is governed by the Hamiltonian [7]:

$$H(\mathbf{x}, \mathbf{p}; \theta) = \frac{R\mathbf{p}^2}{2} + \mathcal{U}(\mathbf{x}; \theta), \tag{2.1}$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{p} = (p_1, p_2, p_3)$ and θ is the azimuthal angle, commonly used in accelerator theory as an independent variable playing the role of time. The quantity R in equation (2.1) denotes the mean radius of the machine. The variables x_3, p_3 constitute a canonical conjugate pair, responsible for the longitudinal motion of the beam

$$x_3 = -\frac{\sigma}{\sqrt{|\mathcal{K}|}} sign(\mathcal{K}) \qquad ; \qquad p_3 = -\eta\sqrt{|\mathcal{K}|} sign(\mathcal{K}), \tag{2.2}$$

where η is the deviation of the actual energy E of the particle under consideration with respect to the energy E_s of the synchronous particle

$$\eta = \frac{1}{\beta_s^2} \frac{E - E_s}{E_s} \qquad ; \qquad \left(\beta_s = \frac{v_s}{c}\right) \tag{2.3}$$

and σ is the displacement of the longitudinal position of the particle with respect to the longitudinal position of the synchronous particle

$$\sigma = s - \beta_s ct. \tag{2.4}$$

The quantity $\mathcal{K}$ is the auto-phasing coefficient (phase slip factor), related to the momentum compaction factor α_M through the equation

$$\mathcal{K} = \alpha_M - \frac{1}{\gamma_s^2} \qquad ; \qquad \left(\gamma_s = \frac{1}{\sqrt{1 - \beta_s^2}}\right). \tag{2.5}$$

The beam propagation in the plane transversal to the particle orbit is described by the canonical conjugate pairs

$$x_k = \widetilde{x}_k - \eta D_k(\theta) \qquad ; \qquad p_k = \widetilde{p}_k - \frac{\eta}{R}\dot{D}_k(\theta) \qquad (k = 1, 2). \tag{2.6}$$

In equation (2.6) $\widetilde{x}_k$ is the actual position of our particle in the transversal plane and $\widetilde{p}_k$ is the canonical conjugate momentum scaled by the total momentum $p_s = m_0\beta_s\gamma_s c$ of the synchronous particle. The function $D_k(\theta)$ is the dispersion function defined as a solution of the equation

$$\ddot{D}_k(\theta) + G_k(\theta)D_k(\theta) = R^2 K_k(\theta), \tag{2.7}$$

where $G_k(\theta)$ are the focusing strengths of the linear machine in the two transverse directions, $K_k(\theta)$ is the local curvature of the orbit and the dot [as well as in equation (2.6)] stands for differentiation with respect to θ.

The potential function $\mathcal{U}(\mathbf{x};\theta)$ in equation (2.1) consists of two parts:

$$\mathcal{U}(\mathbf{x};\theta) = \mathcal{U}_b(x_1, x_2;\theta) + \mathcal{U}_s(x_3;\theta), \tag{2.8}$$

where $\mathcal{U}_b(x_1, x_2;\theta)$ describes the transverse motion (betatron motion) and is given by

$$\mathcal{U}_b(x_1, x_2;\theta) = \frac{1}{2R}\Big[G_1(\theta)x_1^2 + G_2(\theta)x_2^2\Big] + \mathcal{V}(x_1, x_2;\theta), \tag{2.9}$$

while $\mathcal{U}_s(x_3;\theta)$ is responsible for the longitudinal motion (synchrotron motion) and has the form:

$$\mathcal{U}_s(x_3;\theta) = -sign(\mathcal{K})\frac{\Delta E_0}{\beta_s E_s}\frac{c}{2\pi\omega}\cos\left[\frac{\omega\sqrt{|\mathcal{K}|}sign(\mathcal{K})}{\beta_s c}x_3 + \varphi_0\right]. \tag{2.10}$$

In formula (2.10) ΔE_0 is the maximum energy gain per turn, ω and φ_0 being the angular frequency and phase of the accelerating voltage respectively.

From classical mechanics it is well-known that the Hamilton-Jacobi equation associated with the Hamiltonian (2.1) is

$$\frac{\partial S(\mathbf{x};\theta)}{\partial\theta} + \frac{R}{2}\mathbf{p}^2(\mathbf{x};\theta) + \mathcal{U}(\mathbf{x};\theta) = 0, \tag{2.11}$$

where

$$\mathbf{p}(\mathbf{x};\theta) = \nabla S(\mathbf{x};\theta). \tag{2.12}$$

For a given arbitrary integral of equation (2.11) a family of trajectories $\mathbf{q}(\theta)$ is generated that solve the first order (vector) differential equation

$$\dot{\mathbf{q}}(\theta) = R\mathbf{p}[\mathbf{q}(\theta);\theta]. \tag{2.13}$$

Moreover the continuous distribution of trajectories with associated density $\varrho(\mathbf{x};\theta)$ obeys the continuity equation

$$\frac{\partial\varrho(\mathbf{x};\theta)}{\partial\theta} + R\nabla\cdot[\varrho(\mathbf{x};\theta)\mathbf{p}(\mathbf{x};\theta)] = 0 \tag{2.14}$$

and in addition, taking the gradient of equation (2.11) we obtain an equation for $\mathbf{p}(\mathbf{x};\theta)$

$$\frac{\partial\mathbf{p}}{\partial\theta} + R(\mathbf{p}\cdot\nabla)\mathbf{p} + \nabla\mathcal{U}(\mathbf{x};\theta) = 0. \tag{2.15}$$

Thus the system (2.14), (2. 15) [or equivalently (2.11) and (2.14)] represents a closed set of equations, describing the Hamilton-Jacobi fluid as a mechanical system living in configuration space [8].

Let us consider the motion of a test particle in the fluid comprised by the rest of the beam. No dissipative forces of Stokes type are present, as soon as we assume the beam fluid to be inviscid. However, the discrete nature of collisions between particles (intra-beam scattering) give rise to kinetic fluctuations in the one particle distribution function. As a consequence the gas-dynamic functions $\varrho(\mathbf{x};\theta)$ and $\mathbf{p}(\mathbf{x};\theta)$ of the Hamilton-Jacobi fluid fluctuate as well. Fluctuating contributions to the one particle distribution function are generated also by the electro-magnetic interaction between particles in the beam. All this means that the beam is a real medium with a finite number of particles within a physically infinitesimal volume and substituting it by a continuous medium is not justified [2]. External noise could be introduced into the beam from the surroundings (RF noise, fluctuations in the parameters of magnetic elements, etc.), which complexifies the physical picture additionally.

Leaving more speculations aside we consider the motion of our test particle in a random inhomogeneous medium and random velocity field. The particle dynamics is governed by the Hamiltonian

$$H(\mathbf{x},\mathbf{p};\theta) = \frac{R\mathbf{p}^2}{2} + \mathbf{p}\cdot\mathbf{z}(\theta) + \mathcal{U}(\mathbf{x};\theta) + \widetilde{\mathcal{U}}(\mathbf{x};\theta), \tag{2.16}$$

where $\mathbf{z}(\theta)$ is a random velocity field with formal correlation properties

$$\langle \mathbf{z}(\theta) \rangle = 0 \qquad ; \qquad \langle z_k(\theta) z_m(\theta_1) \rangle = R\epsilon_{km}\delta(\theta-\theta_1). \tag{2.17}$$

The quantity $\widetilde{\mathcal{U}}(\mathbf{x};\theta)$ is a random potential accounting for the fluctuation of the medium. In what follows we shall consider the random potential field $\widetilde{\mathcal{U}}(\mathbf{x};\theta)$ δ-correlated with zero mean value and correlation function $A(\mathbf{x},\mathbf{x}_1;\theta)$

$$\left\langle \widetilde{\mathcal{U}}(\mathbf{x};\theta) \right\rangle = 0 \qquad ; \qquad \left\langle \widetilde{\mathcal{U}}(\mathbf{x};\theta)\widetilde{\mathcal{U}}(\mathbf{x}_1;\theta_1) \right\rangle = A(\mathbf{x},\mathbf{x}_1;\theta)\delta(\theta-\theta_1). \tag{2.18}$$

In the Hamiltonian (2.16) we have discarded the "constant" term $\mathbf{z}^2(\theta)/2R$ for it does not give contribution to the dynamics. Moreover, ϵ_{km} has the dimension of emittance and we call it thermal beam emittance tensor.

We would like to note that the equation of random trajectories [replacing now equation (2.13)]

$$\dot{\mathbf{q}}(\theta) = R\mathbf{p}[\mathbf{q}(\theta);\theta] + \mathbf{z}(\theta) \tag{2.19}$$

is in fact the equation for the characteristics of

$$\frac{dC}{d\theta} = \frac{\partial C}{\partial \theta} + (R\mathbf{p} + \mathbf{z})\cdot\nabla C = 0 \tag{2.20}$$

describing the mixing of concentrations $C(\mathbf{x};\theta)$ of different species in a random velocity field.

Instead of equations (2.11), (2.14) and (2.15) we have now the system

$$\frac{\partial S}{\partial \theta} + \frac{R}{2}\mathbf{p}^2 + \mathbf{p}\cdot\mathbf{z}(\theta) + \mathcal{U}(\mathbf{x};\theta) + \widetilde{\mathcal{U}}(\mathbf{x};\theta) = 0, \tag{2.21a}$$

$$\frac{\partial \varrho}{\partial \theta} + \nabla\cdot\{[R\mathbf{p} + \mathbf{z}(\theta)]\varrho\} = 0, \tag{2.21b}$$

$$\frac{\partial \mathbf{p}}{\partial \theta} + [(R\mathbf{p} + \mathbf{z}) \cdot \nabla]\mathbf{p} + \nabla \mathcal{U}(\mathbf{x}; \theta) + \nabla \widetilde{\mathcal{U}}(\mathbf{x}; \theta) = 0, \tag{2.21c}$$

which specifies the evolution law of the Hamilton-Jacobi fluid with hydrodynamic Langevin sources.

3 Kinetic Equation for the One-Point Probability Density

We define a joint probability density for fixed values of the classical action $S(\mathbf{x}; \theta)$, of the momentum $\mathbf{p}(\mathbf{x}; \theta)$ and the density of random trajectories $\varrho(\mathbf{x}; \theta)$ as

$$W(S, \mathbf{p}, \varrho|\mathbf{x}; \theta) = \langle W_r(S, \mathbf{p}, \varrho|\mathbf{x}; \theta)\rangle_{\mathbf{z},\widetilde{\mathcal{U}}}, \tag{3.1}$$

where $\langle ... \rangle_{\mathbf{z},\widetilde{\mathcal{U}}}$ denotes statistical average over the ensemble of realizations of the stochastic processes indicated.

Note that now $S(\mathbf{x}; \theta)$, $\mathbf{p}(\mathbf{x}; \theta)$ and $\varrho(\mathbf{x}; \theta)$ are random functions [more precisely, functionals of the random velocity field $\mathbf{z}(\theta)$ and the random potential $\widetilde{\mathcal{U}}(\mathbf{x}; \theta)$] according to the system (2.21). A closed kinetic equation for the one-point probability density taking into account the gas-dynamic equations (2.21) can be derived by particular choice of the dependence of W_r on the density $\varrho(\mathbf{x}; \theta)$ [11], that is

$$W_r(S, \mathbf{p}, \varrho|\mathbf{x}; \theta) = \varrho(\mathbf{x}; \theta)\delta[S(\mathbf{x}; \theta) - S]\delta[\mathbf{p}(\mathbf{x}; \theta) - \mathbf{p}]. \tag{3.2}$$

Differentiating equation (3.1) with respect to "time" θ and using the gas-dynamic equations (2.21) it is straightforward to obtain the following kinetic equation

$$\frac{\partial W}{\partial \theta} + R\mathbf{p} \cdot \nabla W + \left(\frac{R\mathbf{p}^2}{2} - \mathcal{U}\right)\frac{\partial W}{\partial S} - \nabla \mathcal{U} \cdot \nabla_p W =$$

$$= -\nabla \cdot \langle \mathbf{z} W_r \rangle + \frac{\partial}{\partial S}\left\langle \widetilde{\mathcal{U}} W_r \right\rangle + \nabla_p \cdot \left\langle W_r \nabla \widetilde{\mathcal{U}} \right\rangle. \tag{3.3}$$

It is worthwhile to note that if we let W_r depend on $\varrho(\mathbf{x}; \theta)$ through a generic function it will turn out that the only possibility to cancel terms proportional to $\nabla \cdot \mathbf{p}$ appearing in equation (3.3) is to allow W_r be a linear function of $\varrho(\mathbf{x}; \theta)$. However, the kinetic equation for the one-point probability density (3.1) with an arbitrary dependence on the density of random trajectories can be found in a closed form if the Hessian matrix

$$\mathcal{H}_{mn} = \frac{\partial^2 S(\mathbf{x}; \theta)}{\partial x_m \partial x_n} \tag{3.4}$$

of the classical action is included in the joint probability density [9], [10] and the system (2.21) is supplemented with an equation for the quantity defined by equation (3.4).

We still have not reached our final goal, since the right hand side of equation (3.3) contains the yet unknown correlators of the random velocity field, the random potential field and W_r.

In order to split the above mentioned correlations let us consider a generic functional $\mathcal{R}[F]$ of the random Gaussian tensor field $F_{k_1,\ldots,k_n}(\mathbf{r};\theta)$. Then the following relation holds [11] - [13]

$$\langle F_{k_1,\ldots,k_n}(\mathbf{r})\mathcal{R}[F]\rangle = \int d^n\mathbf{r}_1 \langle F_{k_1,\ldots,k_n}(\mathbf{r}) F_{m_1,\ldots,m_n}(\mathbf{r}_1)\rangle \left\langle \frac{\delta \mathcal{R}[F]}{\delta F_{m_1,\ldots,m_n}(\mathbf{r}_1)} \right\rangle, \tag{3.5}$$

which is known as the Furutsu-Novikov formula. In (3.5) $\mathbf{r}$ collects all the continuous arguments of the random tensor field, $\delta/\delta F_{m_1,\ldots,m_n}(\mathbf{r})$ denotes the functional derivative with respect to the random field and summation over repeated indices is implied. To apply the Furutsu- Novikov formula (3.5) we need the functional derivatives of $S(\mathbf{x};\theta)$, $\varrho(\mathbf{x};\theta)$ and $\mathbf{p}(\mathbf{x};\theta)$ with respect to the random velocity field $\mathbf{z}(\theta)$ and the random potential $\widetilde{\mathcal{U}}(\mathbf{x};\theta)$. From equations (2.21a-c) it is easy to find

$$\frac{\delta S(\mathbf{x};\theta)}{\delta z_k(\theta)} = -p_k(\mathbf{x};\theta) \quad ; \quad \frac{\delta S(\mathbf{x};\theta)}{\delta \widetilde{\mathcal{U}}(\mathbf{x}_1;\theta)} = -\delta(\mathbf{x}-\mathbf{x}_1) \quad ; \quad \frac{\delta S(\mathbf{x};\theta)}{\delta \nabla_{1k}\widetilde{\mathcal{U}}(\mathbf{x}_1;\theta)} = 0, \tag{3.6a}$$

$$\frac{\delta \varrho(\mathbf{x};\theta)}{\delta z_k(\theta)} = -\frac{\partial \varrho(\mathbf{x};\theta)}{\partial x_k} \quad ; \quad \frac{\delta \varrho(\mathbf{x};\theta)}{\delta \widetilde{\mathcal{U}}(\mathbf{x}_1;\theta)} = 0 \quad ; \quad \frac{\delta \varrho(\mathbf{x};\theta)}{\delta \nabla_{1k}\widetilde{\mathcal{U}}(\mathbf{x}_1;\theta)} = 0, \tag{3.6b}$$

$$\frac{\delta p_m(\mathbf{x};\theta)}{\delta z_k(\theta)} = -\frac{\partial p_m(\mathbf{x};\theta)}{\partial x_k} \quad ; \quad \frac{\delta p_m(\mathbf{x};\theta)}{\delta \widetilde{\mathcal{U}}(\mathbf{x}_1;\theta)} = 0 \quad ; \quad \frac{\delta p_m(\mathbf{x};\theta)}{\delta \nabla_{1k}\widetilde{\mathcal{U}}(\mathbf{x}_1;\theta)} = -\delta_{km}\delta(\mathbf{x}-\mathbf{x}_1). \tag{3.6c}$$

By virtue of (3.6), (2.17) and (2.18) we cast equation (3.3) into the form:

$$\begin{aligned} &\frac{\partial W}{\partial \theta} + R\mathbf{p}\cdot\nabla W + \left(\frac{R\mathbf{p}^2}{2} - \mathcal{U}\right)\frac{\partial W}{\partial S} - \nabla\mathcal{U}\cdot\nabla_p W = \\ &= \frac{R\epsilon_{km}}{2}\nabla_k\nabla_m W - \frac{\mathcal{A}(\theta)}{2}\frac{\partial^2 W}{\partial S^2} + \frac{\mathcal{C}_{km}(\theta)}{2}\nabla_{p_k}\nabla_{p_m} W, \end{aligned} \tag{3.7a}$$

where we have taken into account the expansion of the correlation function (2.18):

$$A(\mathbf{x},\mathbf{y};\theta) = \mathcal{A}(\theta) + \mathcal{B}_k(\theta)(x_k - y_k) + \frac{1}{2}\mathcal{C}_{km}(\theta)(x_k - y_k)(x_m - y_m) + \cdots. \tag{3.8}$$

Without loss of generality the first term in the Taylor expansion (3.8) of the correlation function can be taken equal to zero, since it does not contribute to the dynamics (it embeds the gauge properties of the random potential field and can be scaled to zero). Thus we finally arrive at the desired kinetic equation for the one- point probability density:

$$\begin{aligned} &\frac{\partial W}{\partial \theta} + R\mathbf{p}\cdot\nabla W + \left(\frac{R\mathbf{p}^2}{2} - \mathcal{U}\right)\frac{\partial W}{\partial S} - \nabla\mathcal{U}\cdot\nabla_p W = \\ &= \frac{R\epsilon_{km}}{2}\nabla_k\nabla_m W + \frac{\mathcal{C}_{km}(\theta)}{2}\nabla_{p_k}\nabla_{p_m} W, \end{aligned} \tag{3.7}$$

The kinetic equation (3.7) is rather complicated to be solved directly, so approximate methods to analyze it should be involved. For that purpose let us integrate (3.7) over S, that is exclude the classical action from consideration. We get

$$\frac{\partial w}{\partial \theta} + R\mathbf{p}\cdot\nabla w - \nabla\mathcal{U}\cdot\nabla_p w = \frac{R\epsilon_{km}}{2}\nabla_k\nabla_m w + \frac{\mathcal{C}_{km}(\theta)}{2}\nabla_{p_k}\nabla_{p_m} w, \tag{3.9}$$

where

$$w(\mathbf{p},\varrho|\mathbf{x};\theta) = \int dSW(S,\mathbf{p},\varrho|\mathbf{x};\theta). \tag{3.10}$$

If we further integrate equation (3.9) over $\mathbf{p}$ we obtain

$$\frac{\partial\langle\varrho\rangle}{\partial\theta} + \nabla\cdot\left[\langle\varrho\rangle\mathbf{v}_{(+)}\right] - \frac{R\epsilon_{km}}{2}\nabla_k\nabla_m\langle\varrho\rangle = 0, \tag{3.11}$$

where

$$\langle\varrho(\mathbf{x};\theta)\rangle = \int d\mathbf{p}w(\mathbf{p},\varrho|\mathbf{x};\theta), \tag{3.12a}$$

$$\langle\varrho(\mathbf{x};\theta)\rangle\mathbf{v}_{(+)}(\mathbf{x};\theta) = R\int d\mathbf{p}\mathbf{p}w(\mathbf{p},\varrho|\mathbf{x};\theta). \tag{3.12b}$$

Defining the osmotic velocity $\mathbf{u}(\mathbf{x};\theta)$ according to the Fick's law

$$\langle\varrho(\mathbf{x};\theta)\rangle u_k(\mathbf{x};\theta) = -\frac{R\epsilon_{km}}{2}\nabla_m\langle\varrho(\mathbf{x};\theta)\rangle \tag{3.13}$$

one can write (3.11) in the form of a continuity equation

$$\frac{\partial\langle\varrho\rangle}{\partial\theta} + \nabla\cdot(\langle\varrho\rangle\mathbf{v}) = 0, \tag{3.14}$$

where

$$\mathbf{v}(\mathbf{x};\theta) = \mathbf{v}_{(+)}(\mathbf{x};\theta) + \mathbf{u}(\mathbf{x};\theta) \tag{3.15}$$

is the current velocity. Next we introduce the stress tensor [2]

$$\Pi_{mn}(\mathbf{x};\theta) = R^2\int d\mathbf{p}p_m p_n w(\mathbf{p},\varrho|\mathbf{x};\theta), \tag{3.16}$$

which consists of two parts

$$\Pi_{mn} = \langle\varrho\rangle v_{(+)m}v_{(+)n} + \mathcal{G}_{mn}. \tag{3.17}$$

The second term in equation (3.17)

$$\mathcal{G}_{mn}(\mathbf{x};\theta) = \int d\mathbf{p}\Big[Rp_m - v_{(+)m}\Big]\Big[Rp_n - v_{(+)n}\Big]w(\mathbf{p},\varrho|\mathbf{x};\theta) \tag{3.18}$$

is called the internal stress tensor. Multiplying the kinetic equation (3.9) by $R\mathbf{p}$ and integrating over $\mathbf{p}$ we obtain the transport equation for the momentum density

$$\frac{\partial\Big[\langle\varrho\rangle v_{(+)n}\Big]}{\partial\theta}+\nabla_k\Big[\langle\varrho\rangle v_{(+)k}v_{(+)n}\Big]-\frac{R\epsilon_{km}}{2}\nabla_k\nabla_m\Big[\langle\varrho\rangle v_{(+)n}\Big]=-R\langle\varrho\rangle\nabla_n\mathcal{U}-\nabla_k\mathcal{G}_{kn}, \quad (3.19)$$

or in alternative form

$$\frac{\partial v_{(+)n}}{\partial\theta}+\Big[\mathbf{v}_{(-)}\cdot\nabla\Big]v_{(+)n}-\frac{R\epsilon_{km}}{2}\nabla_k\nabla_m v_{(+)n}=-R\nabla_n\mathcal{U}-\frac{1}{\langle\varrho\rangle}\nabla_k\mathcal{G}_{kn}, \quad (3.20a)$$

where use has been made of equations (3.16-18) and (3.11), and the backward velocity field

$$\mathbf{v}_{(-)}(\mathbf{x};\theta)=\mathbf{v}(\mathbf{x};\theta)+\mathbf{u}(\mathbf{x};\theta) \quad (3.21)$$

has been introduced. One can immediately recognize in the left hand side of equation (3.20a) the mean backward derivative [8, 14, 15] of the forward velocity

$$\mathcal{D}_{(-)}v_{(+)n}(\mathbf{x};\theta)=-R\nabla_n\mathcal{U}-\frac{1}{\langle\varrho\rangle}\nabla_k\mathcal{G}_{kn}. \quad (3.20)$$

Perform now "time" inversion transformation in equation (3.20) according to the relations [8]:

$$\theta\longrightarrow\theta'=-\theta \qquad ; \qquad \mathbf{x}(\theta)\longrightarrow\mathbf{x}'(\theta')=\mathbf{x}(\theta) \qquad ; \qquad \mathbf{v}(\theta)\longrightarrow\mathbf{v}'(\theta')=-\mathbf{v}(\theta). \quad (3.22)$$

As a consequence of (3.22) one has

$$\frac{\partial}{\partial\theta'}=-\frac{\partial}{\partial\theta} \qquad ; \qquad \nabla_{x'}=\nabla_x \qquad ; \qquad \nabla_{v'}=-\nabla_v. \quad (3.23a)$$

In addition the forward and backward velocities and mean derivatives are transformed as follows [8]

$$\mathbf{v}_{(\pm)}(\mathbf{x};\theta)\longrightarrow\mathbf{v}'_{(\pm)}(\mathbf{x}';\theta')=-\mathbf{v}_{(\mp)}(\mathbf{x};\theta), \quad (3.23b)$$

$$\mathcal{D}_{(\pm)}f'(\mathbf{x}';\theta')=-\mathcal{D}_{(\mp)}f(\mathbf{x};\theta), \quad (3.23c)$$

where $f(\mathbf{x};\theta)$ is a generic function. Since the internal stress tensor $\mathcal{G}_{kn}$ is a dynamic characteristic of motion under time inversion its divergence changes sign. This also follows from the particular form of the "collision integral" [the right hand side of the kinetic equation (3.7)]. Therefore from (3.20) with (3.22) and (3.23) in hand we obtain

$$\mathcal{D}_{(+)}v_{(-)n}(\mathbf{x};\theta)=-R\nabla_n\mathcal{U}+\frac{1}{\langle\varrho\rangle}\nabla_k\mathcal{G}_{kn}. \quad (3.24)$$

Equations (3.20) and (3.24) provide us two opportunities. First, summing them up we express the transport equation for the momentum density in terms of the current and osmotic velocities as

$$\frac{\partial \mathbf{v}}{\partial \theta} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -R\nabla\mathcal{U} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{R\epsilon_{km}}{2}\nabla_k\nabla_m\mathbf{u}. \tag{3.25}$$

This is nothing else but the Nelson's stochastic generalization of Newton's law [8, 14, 15]. Secondly, subtracting equations (3.20) and (3.24) we obtain an equation for the internal stress tensor to be determined, that is:

$$\frac{\partial u_n}{\partial \theta} + (\mathbf{v} \cdot \nabla)u_n - (\mathbf{u} \cdot \nabla)v_n + \frac{R\epsilon_{km}}{2}\nabla_k\nabla_m v_n = \frac{1}{\langle \varrho \rangle}\nabla_k\mathcal{G}_{kn}. \tag{3.26}$$

In the isotropic case $\epsilon_{km} = \epsilon\delta_{km}$ (see the next section) by virtue of the equation

$$\frac{\partial \mathbf{u}_1}{\partial \theta} + \nabla_1(\mathbf{u}_1 \cdot \mathbf{v}_1) = \frac{R\epsilon}{2}\nabla_1(\nabla_1 \cdot \mathbf{v}_1) \tag{3.27}$$

relating the current and osmotic velocity (which is a direct consequence of Fick's law and the continuity equation) we arrive at the following expression for the internal stress tensor:

$$\mathcal{G}^{(1)}_{kn} = \frac{R\epsilon}{2}\langle \varrho \rangle(\nabla_{1k}v_{1n} + \nabla_{1n}v_{1k}), \tag{3.28}$$

where [compare with equation (4.2a)]

$$\mathcal{G}^{(1)}_{kl} = \left(\widehat{\mathcal{M}}\widehat{\mathcal{G}}\widehat{\mathcal{M}}^T\right)_{kl} = \mathcal{M}_{km}\mathcal{M}_{ln}\mathcal{G}_{mn}. \tag{3.29}$$

Transforming back (3.28) to the original coordinates we obtain

$$\mathcal{G}_{kn} = \frac{R\langle \varrho \rangle}{2}(\epsilon_{km}\nabla_m v_n + \epsilon_{nm}\nabla_k v_m). \tag{3.30}$$

Resuming the results of the present section it should be mentioned that the continuity equation (3.11) and the transport equation for the momentum density (3.19) are equivalent to Nelson's scheme of stochastic mechanics. Let us also note that the characteristics of the fluctuating beam medium, embedded in the random potential (2.18) do not enter the simple hydrodynamic approximation procedure adopted here up to the second moment. It remains, however to analyze the corrections to the evolution law of the Madelung fluid by taking into account the balance equation for the kinetic energy density. This can be done by employing more complete closure techniques to accomplish the transition between kinetic and hydrodynamic description.

4 The Schroedinger-Like Equation

Our starting point is the system of partial differential equations (3.13), (3.14) and (3.25) derived in the preceding section, which in fact represents the set of equations describing the evolution of the Madelung fluid in stochastic mechanics [8, 15]. Following [6] we perform a coordinate transformation

$$\mathbf{x}_1 = \widehat{\mathcal{M}}\mathbf{x} \qquad (x_{1n} = \mathcal{M}_{nm}x_m), \tag{4.1}$$

such that the transformed emittance tensor

$$\epsilon'_{kl} = \left(\widehat{\mathcal{M}}\widehat{\epsilon}\widehat{\mathcal{M}}^T\right)_{kl} = \mathcal{M}_{km}\mathcal{M}_{ln}\epsilon_{mn} \tag{4.2a}$$

is proportional to the unit tensor δ_{kl}

$$\epsilon'_{kl} = \epsilon\delta_{kl} \tag{4.2b}$$

by a factor ϵ, where $(\cdots)^T$ denotes matrix transposition. The scaling factor can be chosen any of the eigenvalues ϵ_k $(k = 1, 2, 3)$ of the original emittance tensor ϵ_{kl}. Provided ϵ_{kl} is symmetric, the matrix $\widehat{\mathcal{M}}$ has the following structure

$$\widehat{\mathcal{M}} = \widehat{\mathcal{A}}\widehat{\mathcal{O}}, \tag{4.3}$$

where $\widehat{\mathcal{O}}$ is an orthogonal matrix and $\widehat{\mathcal{A}}$ is the diagonal matrix

$$\mathcal{A}_{kl} = \sqrt{\frac{\epsilon}{\epsilon_k}}\delta_{kl}. \tag{4.4}$$

Furthermore, the transformed current and osmotic velocities are [16]

$$\mathbf{v}_1 = \widehat{\mathcal{M}}\mathbf{v} \qquad ; \qquad \mathbf{u}_1 = \widehat{\mathcal{M}}\mathbf{u}, \tag{4.5a}$$

while the probability density in the new random coordinates is

$$\varrho_1(\mathbf{x}_1;\theta) = \left|\det\widehat{\mathcal{M}}\right|^{-1}\langle\varrho(\mathbf{x};\theta)\rangle. \tag{4.5b}$$

Then the transformed equations of stochastic mechanics read as

$$\frac{\partial\varrho_1}{\partial\theta} + \nabla_1\cdot(\varrho_1\mathbf{v}_1) = 0, \tag{4.6a}$$

$$\varrho_1\mathbf{u}_1 = -\frac{R\epsilon}{2}\nabla_1\varrho_1, \tag{4.6b}$$

$$\frac{\partial\mathbf{v}_1}{\partial\theta} + (\mathbf{v}_1\cdot\nabla_1)\mathbf{v}_1 = -R\epsilon\nabla_\epsilon\mathcal{U} + (\mathbf{u}_1\cdot\nabla_1)\mathbf{u}_1 - \frac{R\epsilon}{2}\nabla_1^2\mathbf{u}_1, \tag{4.6c}$$

where

$$(\nabla_\epsilon)_n = \frac{1}{\epsilon_n}\nabla_{1n}. \tag{4.7}$$

We are looking now for a Schroedinger-like equation of the type

$$iR\epsilon\frac{\partial\psi}{\partial\theta} = \widehat{\mathbf{H}}\psi \tag{4.8}$$

equivalent to the system (4.6) through the well-known de Broglie ansatz

$$\psi(\mathbf{x}_1;\theta) = \sqrt{\varrho_1(\mathbf{x}_1;\theta)}\exp\left[\frac{i}{R\epsilon}\mathcal{S}(\mathbf{x}_1;\theta)\right], \tag{4.9}$$

where $\widehat{\mathbf{H}}$ is a second order differential operator with known (constant) coefficients in front of the second derivatives. The basic requirement we impose on the operator $\widehat{\mathbf{H}}$ is to be Hermitian

$$\int d\mathbf{x}_1 \psi_1^* \widehat{\mathbf{H}} \psi_2 = \int d\mathbf{x}_1 \psi_2 \widehat{\mathbf{H}}^* \psi_1^*, \tag{4.10}$$

which defines it (as can be easily shown) up to a generic scalar and vector functions. Without loss of generality one can write

$$\widehat{\mathbf{H}} = \frac{1}{2}[iR\epsilon\nabla_1 + \mathbf{A}(\mathbf{x}_1;\theta)]^2 + \Phi(\mathbf{x}_1;\theta), \tag{4.11}$$

where the vector function $\mathbf{A}(\mathbf{x}_1;\theta)$ and the scalar function $\Phi(\mathbf{x}_1;\theta)$ define some effective electro-magnetic field. Substitution of the ansatz (4.9) into equation (4.8) followed by separation of terms by real and imaginary part yields:

$$\mathbf{v}_1 = \nabla_1 \mathcal{S} - \mathbf{A}, \tag{4.12a}$$

$$\frac{\partial \mathbf{v}_1}{\partial \theta} + (\mathbf{v}_1 \cdot \nabla_1)\mathbf{v}_1 = \mathbf{E} + \mathbf{v}_1 \times \mathbf{B} + (\mathbf{u}_1 \cdot \nabla_1)\mathbf{u}_1 - \frac{R\epsilon}{2}\nabla_1^2 \mathbf{u}_1, \tag{4.12b}$$

where [17]

$$\mathbf{E} = -\nabla_1 \Phi - \frac{\partial \mathbf{A}}{\partial \theta} \qquad ; \qquad \mathbf{B} = \nabla_1 \times \mathbf{A}. \tag{4.13}$$

Comparing equation (4.12b) with equation (4.6c) we conclude that the transformed external force $-R\epsilon\nabla_\epsilon \mathcal{U}$ equals the force produced by the effective electro-magnetic field:

$$-R\epsilon\nabla_\epsilon \mathcal{U} = \mathbf{E} + \mathbf{v}_1 \times \mathbf{B}. \tag{4.14}$$

In order to find the electro-magnetic potentials $\mathbf{A}(\mathbf{x}_1;\theta)$ and $\Phi(\mathbf{x}_1;\theta)$ we note that the Schroedinger-like equation (4.8) is gauge invariant under local phase transformation

$$\psi_1(\mathbf{x}_1;\theta) = \psi(\mathbf{x}_1;\theta) \exp\left[\frac{i}{R\epsilon}\mathcal{S}_1(\mathbf{x}_1;\theta)\right]. \tag{4.15}$$

This implies that the equation for the new wave function $\psi_1(\mathbf{x}_1;\theta)$ has the same structure as (4.8) with

$$\mathbf{A}(\mathbf{x}_1;\theta) \longrightarrow \mathbf{A}_1(\mathbf{x}_1;\theta) = \mathbf{A}(\mathbf{x}_1;\theta) + \nabla_1 \mathcal{S}_1(\mathbf{x}_1;\theta), \tag{4.16a}$$

$$\Phi(\mathbf{x}_1;\theta) \longrightarrow \Phi_1(\mathbf{x}_1;\theta) = \Phi(\mathbf{x}_1;\theta) - \frac{\partial \mathcal{S}_1(\mathbf{x}_1;\theta)}{\partial \theta}. \tag{4.16b}$$

Moreover, equation (4.14) written in the form

$$-R\epsilon\nabla_\epsilon \mathcal{U} + \nabla_1 \Phi = -\frac{\partial \mathbf{A}}{\partial \theta} + (\nabla_1 \mathcal{S} - \mathbf{A}) \times (\nabla_1 \times \mathbf{A}) \tag{4.17}$$

is gauge invariant under (4.15) with

$$\mathcal{S}(\mathbf{x}_1;\theta) \longrightarrow \mathcal{S}'(\mathbf{x}_1;\theta) = \mathcal{S}(\mathbf{x}_1;\theta) + \mathcal{S}_1(\mathbf{x}_1;\theta). \tag{4.18}$$

Choosing $\mathcal{S}_1(\mathbf{x}_1;\theta) = -\mathcal{S}(\mathbf{x}_1;\theta)$ we obtain the Euler equation

$$-R\epsilon\nabla_\epsilon\mathcal{U} + \nabla_1\Phi_1 = -\frac{\partial \mathbf{A}_1}{\partial\theta} - \mathbf{A}_1 \times (\nabla_1 \times \mathbf{A}_1) \tag{4.19}$$

for the gauge electro-magnetic potentials $\mathbf{A}_1(\mathbf{x}_1;\theta)$ and $\Phi_1(\mathbf{x}_1;\theta)$.

According to (2.8) - (2.10) the external potential $\mathcal{U}(\mathbf{x};\theta)$ entering the Hamiltonian (2.1) can be specified as

$$\mathcal{U}(\mathbf{x};\theta) = \mathcal{U}_0(\mathbf{x};\theta) + \mathcal{V}(\mathbf{x};\theta). \tag{4.20}$$

The term $\mathcal{U}_0(\mathbf{x};\theta)$ governs the linear motion and can be written in the form

$$\mathcal{U}_0(\mathbf{x};\theta) = \frac{1}{2}\mathbf{x}^T\widehat{\mathbf{G}}(\theta)\mathbf{x}, \tag{4.21}$$

where the matrix $\widehat{\mathbf{G}}(\theta)$ is symmetric in general, while $\mathcal{V}(\mathbf{x};\theta)$ is a sum of all nonlinear terms. Further, we split the electric potential $\Phi_1(\mathbf{x}_1;\theta)$ into two parts according to the relation

$$\Phi_1(\mathbf{x}_1;\theta) = \Phi_0(\mathbf{x}_1;\theta) + \Phi_1'(\mathbf{x}_1;\theta), \tag{4.22}$$

where

$$\Phi_0(\mathbf{x}_1;\theta) = \frac{R}{2}\mathbf{x}_1^T\widehat{\mathbf{G}}_1(\theta)\mathbf{x}_1 \qquad ; \qquad \widehat{\mathbf{G}}_1(\theta) = \widehat{\mathcal{M}}\widehat{\mathbf{G}}(\theta)\widehat{\mathcal{M}}^{-1}, \tag{4.23a}$$

$$\Phi_1'(\mathbf{x}_1;\theta) = -\frac{1}{2}\mathbf{A}_1^2(\mathbf{x}_1;\theta). \tag{4.23b}$$

Equation (4.19) takes now the form

$$-R\epsilon\nabla_\epsilon\mathcal{V} = -\frac{\partial \mathbf{A}_1}{\partial\theta} + (\mathbf{A}_1 \cdot \nabla_1)\mathbf{A}_1, \tag{4.24}$$

which transformed back to the original coordinates $\mathbf{x}$ reads as

$$-R\nabla\mathcal{V} = -\frac{\partial \mathbf{A}'}{\partial\theta} + (\mathbf{A}' \cdot \nabla)\mathbf{A}' \qquad \left(\mathbf{A}' = \widehat{\mathcal{M}}\mathbf{A}_1\right). \tag{4.25}$$

Equation (4.12a) suggests an alternative interpretation of the vector potential $\mathbf{A}(\mathbf{x}_1;\theta)$ as the vortex part of the current velocity $\mathbf{v}_1(\mathbf{x}_1;\theta)$. Taking into account (4.16a) and the particular choice of the gauge phase $\mathcal{S}_1(\mathbf{x}_1;\theta)$ one can expect that $\mathbf{A}'(\mathbf{x};\theta)$ will be vortex-free in the original coordinates

$$\mathbf{A}'(\mathbf{x};\theta) = -R\nabla\varphi(\mathbf{x};\theta), \tag{4.26}$$

where $\varphi(\mathbf{x};\theta)$ is the velocity potential [18]. The first integral of the equation (4.25) is then

$$\frac{\partial \varphi(\mathbf{x};\theta)}{\partial \theta} + \frac{R}{2}[\nabla \varphi(\mathbf{x};\theta)]^2 + \mathcal{V}(\mathbf{x};\theta) = g(\theta). \tag{4.27}$$

Without loss of generality the generic function $g(\theta)$ may be set equal to zero as a result of the uncertainty in the definition of the velocity potential (4.26). Equation (4.27) is noting else but the Hamilton-Jacobi equation (2.11) for the "classical action" $\varphi(\mathbf{x};\theta)$, associated with the nonlinear part $\mathcal{V}(\mathbf{x};\theta)$ of the external potential $\mathcal{U}(\mathbf{x};\theta)$ only.

Performing a second [similar to (4.15)] phase transformation according to

$$\psi_2(\mathbf{x}_1;\theta) = \psi_1(\mathbf{x}_1;\theta) \exp\left[\frac{i}{R\epsilon} R\varphi(\mathbf{x}_1;\theta)\right]. \tag{4.28}$$

we obtain the gauge potentials $\mathbf{A}_2(\mathbf{x}_1;\theta)$ and $\Phi_2(\mathbf{x}_1;\theta)$ entering the Schroedinger equation for the wave function $\psi_2(\mathbf{x}_1;\theta)$. They are:

$$\mathbf{A}_2(\mathbf{x}_1;\theta) = R\left(\widehat{\mathbf{I}} - \widehat{\mathcal{A}}^2\right)\nabla_1 \varphi\left(\widehat{\mathcal{M}}^{-1}\mathbf{x}_1;\theta\right), \tag{4.29a}$$

$$\Phi_2(\mathbf{x}_1;\theta) = \Phi_0(\mathbf{x}_1;\theta) + R\mathcal{V}\left(\widehat{\mathcal{M}}^{-1}\mathbf{x}_1;\theta\right) +$$

$$+\frac{R^2}{2}\left[\nabla_1 \varphi\left(\widehat{\mathcal{M}}^{-1}\mathbf{x}_1;\theta\right)\right]^T \widehat{\mathcal{A}}^2\left(\widehat{\mathbf{I}} - \widehat{\mathcal{A}}^2\right)\nabla_1 \varphi\left(\widehat{\mathcal{M}}^{-1}\mathbf{x}_1;\theta\right). \tag{4.29b}$$

The Schroedinger-like equation called upon to replace equation (4.8) reads as

$$iR\epsilon \frac{\partial \psi_2(\mathbf{x}_1;\theta)}{\partial \theta} = \frac{1}{2}[iR\epsilon\nabla_1 + \mathbf{A}_2(\mathbf{x}_1;\theta)]^2 \psi_2(\mathbf{x}_1;\theta) + \Phi_2(\mathbf{x}_1;\theta)\psi_2(\mathbf{x}_1;\theta). \tag{4.30}$$

Retracing the sequence of phase transformations (4.15) and (4.28) we find

$$\psi(\mathbf{x}_1;\theta) = \psi_2(\mathbf{x}_1;\theta) \exp\left\{\frac{i}{R\epsilon}\left[\mathcal{S}(\mathbf{x}_1;\theta) - R\varphi\left(\widehat{\mathcal{M}}^{-1}\mathbf{x}_1;\theta\right)\right]\right\}, \tag{4.31a}$$

$$\mathbf{A}_2(\mathbf{x}_1;\theta) = \mathbf{A}(\mathbf{x}_1;\theta) - \nabla_1\left[\mathcal{S}(\mathbf{x}_1;\theta) - R\varphi\left(\widehat{\mathcal{M}}^{-1}\mathbf{x}_1;\theta\right)\right], \tag{4.31b}$$

$$\Phi_2(\mathbf{x}_1;\theta) = \Phi(\mathbf{x}_1;\theta) + \frac{\partial}{\partial \theta}\left[\mathcal{S}(\mathbf{x}_1;\theta) - R\varphi\left(\widehat{\mathcal{M}}^{-1}\mathbf{x}_1;\theta\right)\right]. \tag{4.31c}$$

The relations (4.31) indicate the equivalence of the Schroedinger-like equations (4.8) and (4.30) up to a global phase transformation, defined by the constant in the coordinates and time phase

$$\mathcal{C} = \mathcal{S}(\mathbf{x}_1;\theta) - R\varphi\left(\widehat{\mathcal{M}}^{-1}\mathbf{x}_1;\theta\right) = const. \tag{4.32}$$

The anisotropy of the random velocity field (2.17) reflects on the appearance of the gauge electro-magnetic potentials. There are two cases in which the vector potential $\mathbf{A}_2(\mathbf{x}_1;\theta)$ vanishes and the scalar potential $\Phi_2(\mathbf{x}_1;\theta)$ is equal (up to a non essential factor R) to the external potential $\mathcal{U}(\mathbf{x}_1;\theta)$. The first case is when the external potential is the harmonic oscillator potential ($\mathcal{V} = 0$) with generally time-dependent frequency, while the second is the isotropic case ($\epsilon_{km} = \epsilon\delta_{km}$).

5 Concluding Remarks

In the present paper we have studied the motion of a test particle in a random inhomogeneous medium comprised by the rest of the beam. As a result of the investigation performed we have shown that Nelson's scheme of stochastic mechanics for particle beams in the case of zero friction, is equivalent to hydrodynamic approximation in the kinetic equation for the one particle distribution function up to the second moment. Further, it has been pointed out that the hydrodynamic equations of continuity and momentum density can be transformed by a change of coordinates and dependent variables into a Schroedinger-like equation. Regardless of the type of the external forces one need to introduce a gauge electro-magnetic field. If the beam constitutes an isotropic medium (holding in the case of symmetric beams) the gauge vector potential vanishes and as a consequence the scalar potential is equal to the potential that accounts for the external force.

The gauge transformation (4.16) is the well-known transformation in classical electromagnetic theory [17] introduced by Weyl, indicating a transition to alternative electromagnetic potentials, which sometimes are easier to find compared to the original ones. Besides that, the transformed potentials define the same electro-magnetic field tensor. Taking into account this fact we have found the gauge electro-magnetic potentials explicitly, depending on the solution of a Hamilton-Jacobi equation for the classical motion of the particle in the anharmonic part of the external potential.

The beam circulating in an accelerator consists of a large number of particles. Obviously, all of them cannot be in the same micro-state. As a result the beam itself generates noise, which plays a role similar to the role of perturbation in stability theory. The essential difference is that here the perturbation is produced by the macroscopic system itself (in addition to the noise introduced from the surroundings). In the present work we have adopted a phenomenological approach to describe beam fluctuations. In this connection it remains to compute the statistical properties of the beam medium in terms of the fluctuation spectrum, which we hope to perform in a forthcoming paper.

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References

[1] L.D. Landau, E.M. Lifshitz, **Statistical Physics**, Pergamon, Oxford, 1957.

[2] Yu.L. Klimontovich, **Statistical Physics**, Harwood Academic Publishers, Chur, 1986.

[3] S. Ichimaru, **Statistical Plasma Physics**, Vol. 1, Addison-Wesley, New York, 1992.

[4] R. Fedele, G. Miele, Nuovo Cimento, Vol. 13D, No. 12 (1991) pp. 1527-1543.

[5] S.I. Tzenov, *The Concept of Stochastic Mechanics in Particle Accelerator Physics*, In **Nonlinear Dynamics in Particle Accelerators: Theory and Experiments**, Arcidosso, Italy 1994, S. Chattopadhyay et al. eds., AIP Conf. Proc. **344**, New York, 1995, pp. 249-258.

[6] S.I. Tzenov, *The Schroedinger-like Equation with Electro-Magnetic Potentials in the Framework of Stochastic Quantization Approach*, INFN/TC - 95/21, INFN Sezione di Napoli (4 August 1995).

[7] S.I. Tzenov, *Resonance Phenomena in Cyclic Accelerators of Charged Particles*, PhD. Thesis, JINR, Dubna, 1991 (Unpublished).

[8] F. Guerra, *Structural Aspects of Stochastic Mechanics and Stochastic Field Theory*, **Physics Reports**, Vol. **77**, No. 3 (1981) pp. 263-312.

[9] A.N. Malakhov, A.I. Saichev, **Soviet Phys. JETP**, Vol. **40**, No. 3 (1975) pp. 467-471.

[10] A.N. Malakhov, A.I. Saichev, **Soviet Phys. JETP**, Vol. **40**, No. 6 (1975) pp. 1032-1035.

[11] V.I. Klyatskin, **Stochastic Equations and Waves in Random Inhomogeneous Media**, Nauka, Moscow, 1980. (in Russian)

[12] K. Furutsu, **Journ. Res. NBS**, Vol. **667D** (1963) p. 303.

[13] E.A. Novikov, **Soviet Phys. JETP**, Vol. **20**, No. 5 (1965) pp. 1290-1294.

[14] E. Nelson, **Physical Review**, Vol. **150**, No. 4 (1966) p. 1079.

[15] Ph. Blanchard, Ph. Combe, W. Zheng, **Mathematical and Physical Aspects of Stochastic Mechanics**, Springer-Verlag, Berlin, 1987.

[16] H. Risken, **The Fokker-Planck Equation**, Springer-Verlag, Berlin, 1984.

[17] M.A. Tonnelat, **The Principles of Electro-Magnetic Theory and Relativity**, Dordrecht, Reidel, 1966.

[18] G.K. Batchelor, **An Introduction to Fluid Dynamics**, Cambridge University Press, Cambridge, 1970.